On trigonometric approximation of functions in the L^p norm

R. N. Mohapatra [1] and B. Szal [2]

[1] University of Central Florida
Orlando, FL 32816, USA
ramm@pegasus.cc.ucf.edu
[2] University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a, Poland
B.Szal@wmie.uz.zgora.pl

Abstract

In this paper we obtain degree of approximation of functions in L^p by operators associated with their Fourier series using integral modulus of continuity. These results generalize many know results and are proved under less stringent conditions on the infinite matrix.

Keywords and phrases: Class $Lip\left(\alpha,p\right)$; Trigonometric approximation; L^p-norm.

2000 Mathematics Subject Classification: 42A10, 41A25.

1 Introduction

Let f be 2π periodic and $f\in L^p\left[0,2\pi\right]$ for $p\geq 1$. Denote by

$$S_n\left(f\right)=S_n\left(f;x\right)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right)\equiv\sum_{k=0}^{n}U_k\left(f;x\right)$$

partial sum of the first $(n+1)$ terms of the Fourier series of $f\in L^p$ $(p\geq 1)$ at a point x, and by

$$\omega_p\left(f;\delta\right)=\sup_{0<|h|\leq\delta}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(x+h\right)-f\left(x\right)\right|^p dx\right\}^{\frac{1}{p}}$$

the integral modulus of continuity of $f \in L^p$. If, for $\alpha > 0$, $\omega_p(f;\delta) = O(\delta^\alpha)$, then we write $f \in Lip(\alpha, p)$ $(p \geq 1)$.

Throughout $\|\cdot\|_{L^p}$ will denote L^p-norm, defined by

$$\|f\|_{L^p} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}} \qquad (f \in L^p(p \geq 1)).$$

In the present paper, we shall consider approximation of $f \in L^p$ by trigonometrical polynomials $T_n(f;x)$, where

$$T_n(f;x) = T_n(f, A; x) := \sum_{k=0}^{n} a_{n,k} S_k(f;x) \qquad (n = 0, 1, 2, ...)$$

and $A := (a_{n,k})$ be a lower triangular infinite matrix of real numbers such that:

$$a_{n,k} \geq 0 \text{ for } k \leq n \text{ and } a_{n,k} = 0 \text{ for } k > n \ \ (k, n = 0, 1, 2....) \tag{1.1}$$

and

$$\sum_{k=0}^{n} a_{n,k} = 1 \, (n = 0, 1, 2...) . \tag{1.2}$$

If $a_{n,k} = \frac{p_k}{P_m}$, where $P_n = p_0 + p_1 + ... + p_n \neq 0$ $(n \geq 0)$, then we shall call this trigonometrical polynomials by

$$R_n(f;x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k(f;x) \qquad (n = 0, 1, 2....) .$$

The case $a_{n,k} = \frac{1}{n+1}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$ of $T_n(f;x)$ yields

$$\sigma_n(f;x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f;x) \qquad (n = 0, 1, 2....) .$$

We shall also use the notations

$$\Delta a_k = a_k - a_{k+1}, \qquad \Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$$

and we shall write $I_1 \ll I_2$ if there exists a positive constant K such that $I_1 \leq K I_2$.

Let $C := (C_n) = \frac{1}{n+1} \sum\limits_{k=0}^{n} c_k$, where $c := (c_n)$ is a sequence of nonnegative numbers. The sequence c is called a nondecreasing (nonincreasing) mean sequence,

briefly $NDMS$ ($NIMS$), if $C \in NDS$ ($C \in NIS$), where NDS (NIS) is the class of nonnegative and nondecreasing (nonincreasing) sequences.

A nonnegative sequence $c := (c_n)$ is called almost monotone decreasing ($AMDS$) (increasing ($AMIS$)) if there exists a constant $K := K(c)$, depending on the sequence c only, such that for all $n \geq m$

$$c_n \leq Kc_m \qquad (Kc_n \geq c_m).$$

Such sequences will be denoted by $c \in AMDS$ and $c \in AMIS$, respectively.

If $C \in AMDS$ ($C \in AMIS$), then we shall say that c is almost monotone decreasing (increasing) mean sequence, briefly $c \in AMDMS$ ($c \in AMIMS$).

When we write that a sequence $(a_{n,k})$ belongs to one of the above classes, it means that it satisfies the required conditions from the above definitions with respect to $k = 0, 1, 2, ..., n$ for all n.

A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called a rest bounded variation sequence (rest bounded variation mean sequence), or briefly $c \in RBVS$ ($c \in RBVMS$), if it has the property

$$\sum_{k=m}^{\infty} |\Delta c_k| \leq K(c)\, c_m \qquad \left(\sum_{k=m}^{\infty} |\Delta C_k| \leq K(c)\, C_m \right) \tag{1.3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_n)$ of nonnegative numbers will be called a head bounded variation sequence (head bounded variation mean sequence), or briefly $c \in HBVS$ ($c \in HBVMS$), if it has the property

$$\sum_{k=0}^{m-1} |\Delta c_k| \leq K(c)\, c_m \qquad \left(\sum_{k=0}^{m-1} |\Delta C_k| \leq K(c)\, C_m \right) \tag{1.4}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only a finite number of nonzero terms and the last nonzero terms is c_N.

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (1.3) or (1.4) for the sequence $\alpha_n := (a_{nk})_{k=0}^{\infty}$.Now we can mention the

conditions to be used later on. Let $A_{n,m} = \frac{1}{m+1}\sum_{k=0}^{m} a_{n,k}$. We assume that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |\Delta_k a_{nk}| \leq K a_{nm} \qquad \left(\sum_{k=m}^{\infty} |\Delta_k A_{nk}| \leq K A_{nm}\right)$$

and

$$\sum_{k=0}^{m-1} |\Delta_k a_{nk}| \leq K a_{nm} \qquad \left(\sum_{k=0}^{m-1} |\Delta_k A_{nk}| \leq K A_{nm}\right)$$

hold if $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$ ($RBVMS$) or $HBVS$ ($HBVMS$), respectively.

It is clear that

$$\begin{aligned} NIS &\subset RBVS \subset AMDS, \\ NIMS &\subset RBVMS \subset AMDMS \end{aligned}$$

and

$$\begin{aligned} NDS &\subset HBVS \subset AMIS, \\ NDMS &\subset HBVMS \subset AMIMS. \end{aligned}$$

In the present paper we shall show that $NIS \subset NIMS$, $AMDS \subset AMDMS$, $NDS \subset NDMS$ and $AMIS \subset AMIMS$, too.

In 1937 E. Quade [8] proved that, if $f \in Lip\,(\alpha, p)$ for $0 < \alpha \leq 1$, then $\|\sigma_n(f) - f\|_{L^p} = O(n^{-a})$ for either $p > 1$ and $0 < \alpha \leq 1$ or $p = 1$ and $0 < \alpha < 1$. He also showed that, if $p = \alpha = 1$, then $\|\sigma_n(f) - f\|_{L^1} = O\left(n^{-1}\log(n+1)\right)$.

There are several generalizations of the above result for $p > 1$ (see, for example[1, 2, 3], [5] and [7]). In [4] P. Chandra extended the work of E. Quade and proved the following theorems:

Theorem 1. *Let $f \in Lip\,(\alpha, p)$ and let (p_n) be positive. Suppose that either*

(*i*) $p > 1$, $0 < \alpha \leq 1$, *and*

(*ii*) $\sum_{k=0}^{n-1} \left|\Delta\left(\frac{P_k}{k+1}\right)\right| = O\left(\frac{P_n}{n+1}\right)$, *or*

(*i*) $p = 1$, $0 < \alpha < 1$, *and*

(*ii*) (p_n) *is nondecreasing and*

$$(n+1)\,p_n = O\left(P_n\right). \tag{1.5}$$

Then

$$\left\|R_n\left(f\right) - f\right\|_{L^p} = O\left(n^{-\alpha}\right).$$

Theorem 2. *Let $f \in Lip\left(1,1\right)$ and let $\left(p_n\right)$ with (1.5) be positive, and that*

$$\left(\left(n+1\right)^{\eta} p_n\right) \in NDS \text{ for some } \eta > 0.$$

Then

$$\left\|R_n\left(f\right) - f\right\|_{L^1} = O\left(n^{-1}\right).$$

In [6] M. Mittal,B. Rhoades,V. Mishra and U. Singh obtained the same degree of approximation as in above theorems, for a more general class of lower triangular matrices, and deduced some of the results of P. Chandra. Namely, they proved the following theorem:

Theorem 3. *Let $f \in Lip\left(\alpha,p\right)$, and let $a_{n,k} \geq 0$ $(k,n = 0,1,...)$, $\left(a_{nk}\right) \in NDS$ or $\left(a_{n,k}\right) \in NIS$ and*

$$\left|\sum_{k=0}^{n} a_{n,k} - 1\right| = O\left(n^{-\alpha}\right).$$

(i) If $p > 1$, $0 < \alpha < 1$, $(n+1)\max\left\{a_{n,0}, a_{n,r}\right\} = O\left(1\right)$, where $r := \left[\frac{n}{2}\right]$, then

$$\left\|T_n\left(f\right) - f\right\|_{L^p} = O\left(n^{-\alpha}\right). \tag{1.6}$$

(ii) If $p > 1$, $\alpha = 1$, then (1.6) is satisfied.

(iii) If $p = 1$, $0 < \alpha < 1$, and $(n+1)\max\left\{a_{n,0}, a_{nn}\right\} = O\left(1\right)$, then (1.6) is satisfied.

In this paper we shall prove that the above mentioned theorems are valid with less stringent assumptions.

2 Statement of the results

Our first theroem deals with a number of embedding results.

Theorem 4. *The following embedding relations are valid:*

(i) $NIS \subset NIMS$,

(ii) $NDS \subset NDMS$,

(iii) $AMDS \subset AMDMS$,

(iv) $AMIS \subset AMIMS$.

Our next theorem deals with degree of convergence of operators involving the infinite matrix.

Theorem 5. *Let* $f \in Lip\,(\alpha, p)$ *and (1.1), (1.2) hold. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* $(a_{n,k}) \in AMIMS$,

(ii) $p > 1$, $0 < \alpha < 1$, $(a_{n,k}) \in AMDMS$ *and* $(n+1)\,a_{n,0} = O\,(1)$,

(iii) $p > 1$, $\alpha = 1$ *and* $\sum\limits_{k=0}^{n-1} |\Delta_k A_{n,k}| = O\left(n^{-1}\right)$,

(iv) $p = 1$, $0 < \alpha < 1$, $\sum\limits_{k=0}^{n-1} |\Delta_k a_{n,k}| = O\left(n^{-1}\right)$ *and* $(n+1)\,a_{n,n} = O\,(1)$,

(v) $p = 1$, $0 < \alpha < 1$, $(a_{n,k}) \in RBVS$ *and* $(n+1)\,a_{n,0} = O\,(1)$,

(vi) $p = \alpha = 1$, $\left((k+1)^{-\beta}\,a_{n,k}\right) \in HBVS$ *for some* $\beta > 0$ *and* $(n+1)\,a_{n,n} = O\,(1)$

maintains, then

$$\|T_n\,(f) - f\|_{L^p} = O\left(n^{-\alpha}\right). \tag{2.1}$$

Remark 1. Let $f \in Lip\,(\alpha, p)$, (1.1) and

$$\left|\sum_{k=0}^{n} a_{n,k} - 1\right| = O\left(n^{-\alpha}\right)$$

hold. Under the assumptions of Theorem 5 $(i) - (vi)$ we can observe that the estimation (2.1) are true, too.

In the special cases, putting $a_{n,k} = \frac{p_k}{P_n}$, where $P_n = p_0 + p_1 + ... + p_n \neq 0$, we can derive from Theorem 5 the following corollary:

Corollary 1. *Let* $f \in Lip\,(\alpha, p)$ *and let* (p_k) *be positive. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* $(p_k) \in AMIMS$,

(ii) $p > 1$, $0 < \alpha < 1$, $(p_k) \in AMDMS$ *and* $(n+1) = O(P_n)$,

(iii) $p > 1$, $\alpha = 1$ *and* $\sum_{k=0}^{n-1} \left| \Delta_k \frac{P_k}{k+1} \right| = O\left(\frac{P_n}{n}\right)$,

(iv) $p = 1$, $0 < \alpha < 1$, $\sum_{k=0}^{n-1} |\Delta_k p_k| = O\left(n^{-1}\right)$ *and* $(n+1)\, p_n = O(P_n)$,

(v) $p = 1$, $0 < \alpha < 1$, $(p_k) \in RBVS$ *and* $(n+1) = O(P_n)$,

(vi) $p = \alpha = 1$, $\left((k+1)^{-\beta} p_k\right) \in HBVS$ *for some* $\beta > 0$ *and* $(n+1)\, p_n = O(P_n)$, *then*

$$\|R_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right).$$

Remark 2. By Theorem 4 we can observe that Theorem 3 and Theorem 1 follow from Remark 1 and Corollary 1 ((*i*), (*iii*)), respectively. Moreover, since $NDC \subset HBVS$, we can derive from Corollary 1 (*vi*) analogous estimate as in Theorem 2 for the deviation $R_n(f) - f$ in the L^p−norm.

3 Auxiliary results

We shall use the following lemmas for the proof of our theorems:

Lemma 1. *[8, Theorem 4] If* $f \in Lip(\alpha, p)$, $p \geq 1$, $0 < \alpha \leq 1$, *then, for any positive integer* n, f *may be approximated in* L^p−*space by a trigonometrical polynomial* t_n *or order* n *such that*

$$\|f - t_n\|_{L^p} = O\left(n^{-\alpha}\right).$$

Lemma 2. *[8, Theorem 5 (i)] If* $f \in Lip(\alpha, 1)$, $0 < \alpha < 1$, *then*

$$\|\sigma_n(f) - f\|_{L^1} = O\left(n^{-\alpha}\right).$$

Lemma 3. *[8, p. 541, last line] If* $f \in Lip(1, p)$ $(p > 1)$, *then*

$$\|\sigma_n(f) - S_n(f)\|_{L^p} = O\left(n^{-1}\right).$$

Lemma 4. *[8, Theorem 6 (i), p 541] Let, for* $0 < \alpha \leq 1$ *and* $p > 1$, $f \in Lip(\alpha, p)$. *Then*

$$\|S_n(f) - f\|_{L^p} = O\left(n^{-\alpha}\right).$$

Lemma 5. *Let (1.1) and (1.2) hold. If* $(a_{n,k}) \in AMIMS$ *or* $(a_{n,k}) \in AMDMS$ *and* $(n+1)\, a_{n,0}$, *then, for* $0 < \alpha < 1$,

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} = O\left((n+1)^{-\alpha}\right)$$

holds.

Proof. Let $r = \left[\frac{n}{2}\right]$. Then, if (1.1) and (1.2) hold,

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} \leq \sum_{k=0}^{r} (k+1)^{-\alpha} a_{n,k} + (r+1)^{-\alpha} \sum_{k=r+1}^{n} a_{n,k}$$

$$\leq \sum_{k=0}^{r} (k+1)^{-\alpha} a_{n,k} + (r+1)^{-\alpha} .$$

By Abel's transformation, we get

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} \leq \sum_{k=0}^{r-1} \left\{ (k+1)^{-\alpha} - (k+2)^{-\alpha} \right\} \sum_{i=0}^{k} a_{n,i}$$

$$+ (r+1)^{-\alpha} \sum_{k=0}^{r} a_{n,k} + (r+1)^{-\alpha} \leq \sum_{k=0}^{r-1} \frac{(k+2)^{\alpha} - (k+1)^{\alpha}}{(k+1)^{\alpha-1} (k+2)^{\alpha}} A_{n,k} + (r+1)^{-\alpha} .$$

Using Lagrange's mean value theorem to the function $f(x) = x^{\alpha}$ $(0 < \alpha < 1)$ on the interval $(k+1, k+2)$ we obtain

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} \leq \sum_{k=0}^{r-1} \frac{\alpha}{(k+2)^{\alpha}} A_{n,k} + (r+1)^{-\alpha} .$$

If $(a_{n,k}) \in AMIMS$, then

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} \ll A_{n,r} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha}} + (r+1)^{-\alpha}$$

$$\ll (r+1)^{-\alpha} \sum_{k=0}^{r} a_{n,k} + (r+1)^{-\alpha} \ll (n+1)^{-\alpha} .$$

When $(a_{n,k}) \in AMDMS$ and $(n+1) a_{n,0} = O(1)$ we get

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} \ll A_{n,0} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha}} + (r+1)^{-\alpha}$$

$$\ll (r+1)^{1-\alpha} a_{n,0} + (r+1)^{-\alpha} \ll (n+1)^{-\alpha} .$$

This completes our proof. $\square$

4 Proofs of the results

4.1 Proof of Theorem 4

(i) If $(a_n) \in NIS$, then

$$(n+2)\sum_{k=0}^{n} a_k = (n+1)\sum_{k=0}^{n+1} a_k + \sum_{k=0}^{n} a_k - (n+1)\,a_{n+1}$$

$$\geq (n+1)\sum_{k=0}^{n+1} a_k + (n+1)\,(a_n - a_{n+1}) \geq (n+1)\sum_{k=0}^{n+1} a_k.$$

Thus

$$\frac{1}{n+2}\sum_{k=0}^{n+1} a_k \leq \frac{1}{n+1}\sum_{k=0}^{n} a_k$$

and $(a_n) \in NIMS$.

(ii) Let $(a_n) \in NDS$. Hence

$$(n+2)\sum_{k=0}^{n} a_k = (n+1)\sum_{k=0}^{n+1} a_k + \sum_{k=0}^{n} a_k - (n+1)\,a_{n+1}$$

$$\leq (n+1)\sum_{k=0}^{n+1} a_k + (n+1)\,(a_n - a_{n+1}) \leq (n+1)\sum_{k=0}^{n+1} a_k.$$

Therefore

$$\frac{1}{n+1}\sum_{k=0}^{n} a_k \leq \frac{1}{n+2}\sum_{k=0}^{n+1} a_k$$

and $(a_n) \in NDMS$.

(iii) Suppose that $(a_n) \in AMDS$ we have for $m \leq l$

$$(l+1)\sum_{i=0}^{m} a_i = (m+1)\sum_{i=0}^{m} a_i + (l-m)\sum_{i=0}^{m} a_i$$

$$\geq (m+1)\left\{\sum_{i=0}^{m} a_i + \frac{1}{K}\,(l-m)\,a_m\right\} \geq (m+1)\left\{\sum_{i=0}^{m} a_i + \frac{1}{K^2}\sum_{i=m+1}^{l} a_i\right\}$$

$$\geq \min\left\{1, \frac{1}{K^2}\right\}(m+1)\sum_{i=0}^{l} a_i.$$

Hence

$$\frac{1}{\min\left\{1, \frac{1}{K^2}\right\}}\frac{1}{m+1}\sum_{i=0}^{m} a_i \geq \frac{1}{l+1}\sum_{i=0}^{l} a_i$$

and $(a_n) \in AMDMS$.

(iv) If $(a_n) \in AMIS$, then for $m \leq l$ we get

$$(l+1)\sum_{i=0}^{m} a_i \leq (m+1)\left\{\sum_{i=0}^{m} a_i + K(l-m)\, a_m\right\}$$

$$\leq (m+1)\left\{\sum_{i=0}^{m} a_i + K^2 \sum_{i=m+1}^{l} a_i\right\} \leq \max\left\{1.K^2\right\}(m+1)\sum_{i=0}^{l} a_i.$$

Thus

$$\frac{1}{m+1}\sum_{i=0}^{m} a_i \leq \max\left\{1.K^2\right\}\frac{1}{l+1}\sum_{i=0}^{l} a_i$$

and $(a_n) \in AMIMS$.

The proof is now complete. $\square$

4.2 Proof of Theorem 5

We prove the cases (i) and (ii) together utilizing Lemmas 4 and 5. Since

$$T_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,k}\left(S_k(f;x) - f(x)\right),$$

thus

$$\left\|T_n(f) - f\right\|_{L^p} \leq \sum_{k=0}^{n} a_{n,k}\left\|S_k(f) - f\right\|_{L^p} \ll \sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} = O\left(n^{-\alpha}\right)$$

and this is (2.1).

Next we consider the case (iii).

Using two times Abel's transformation and (1.2) we get that

$$T_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,k}\left(S_k(f;x) - f(x)\right)$$

$$= \sum_{k=0}^{n-1}\left(S_k(f;x) - S_{k+1}(f;x)\right)\sum_{i=0}^{k} a_{n,i} + S_n(f;x) - f(x)$$

$$= S_n(f;x) - f(x) - \sum_{k=0}^{n-1}(k+1)\, U_{k+1}(f;x)\, A_{n,k}$$

$$= S_n(f;x) - f(x) - \sum_{k=0}^{n-2}(A_{n,k}-A_{n,k+1})\sum_{i=0}^{k}(i+1)U_{i+1}(f;x)$$

$$-A_{n,n-1}\sum_{k=0}^{n-1}(k+1)U_{k+1}(f;x) = S_n(f;x) - f(x)$$

$$-\sum_{k=0}^{n-2}(A_{n,k}-A_{n,k+1})\sum_{i=0}^{k}(i+1)U_{i+1}(f;x) - \frac{1}{n}\sum_{i=0}^{n-1}a_{n,i}\sum_{k=0}^{n-1}(k+1)U_{k+1}(f;x).$$

Hence

$$\|T_n(f)-f\|_{L^p} \le \|S_n(f)-f\|_{L^p}$$

$$+\sum_{k=0}^{n-2}|A_{n,k}-A_{n,k+1}|\left\|\sum_{i=1}^{k+1} iU_i(f)\right\|_{L^p} + \frac{1}{n}\left\|\sum_{k=1}^{n} kU_k(f;x)\right\|_{L^p}. \tag{4.1}$$

Since

$$\sigma_n(f;x) - S_n(f;x) = \frac{1}{n+1}\sum_{k=1}^{n} kU_k(f;x),$$

thus by Lemma 3

$$\left\|\sum_{k=1}^{n} kU_k(f)\right\|_{L^p} = (n+1)\|\sigma_n(f)-S_n(f)\|_{L^p} = O(1). \tag{4.2}$$

By (4.1), (4.2) and Lemma 4 we get that

$$\|T_n(f)-f\|_{L^p} \ll \frac{1}{n} + \sum_{k=0}^{n-1}|A_{n,k}-A_{n,k+1}|.$$

If $\sum\limits_{k=0}^{n-1}|\Delta_k A_{n,k}| = O\left(n^{-1}\right)$, then

$$\|T_n(f)-f\|_{L^p} = O\left(n^{-1}\right)$$

and (2.1) holds.

The cases (iv) and (v) we also prove together.

By Abel's transformation

$$T_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,k}(S_k(f;x)-f(x))$$

$$= \sum_{k=0}^{n-1}(a_{n,k}-a_{n,k+1})\sum_{i=0}^{k}(S_i(f;x)-f(x)) + a_{n,n}\sum_{k=0}^{n}(S_k(f;x)-f(x))$$

$$= \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1}) (k+1) (\sigma_k (f;x) - f(x)) + a_{n,n} (n+1) (\sigma_n (f;x) - f(x)).$$

Using Lemma 2 we get

$$\|T_n (f) - f\|_{L^1} \le \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| (k+1) \|\sigma_k (f) - f\|_{L^1}$$

$$+a_{n,n} (n+1) \|\sigma_n (f) - f\|_{L^1} \ll \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| (k+1)^{1-\alpha}$$

$$+a_{n,n} (n+1)^{1-\alpha} \le (n+1)^{1-\alpha} \left(\sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| + a_{n,n} \right).$$

When the assumptions (iv) hold we get

$$\|T_n (f) - f\|_{L^1} = O\left(n^{-\alpha}\right).$$

If $(a_{n,k}) \in RBVS$, then $(a_{n,k}) \in AMDS$. Thus

$$\|T_n (f) - f\|_{L^1} \ll (n+1)^{1-\alpha} (a_{n,0} + a_{n,n}) \ll (n+1)^{1-\alpha} a_{n,0}.$$

Hence, if $(n+1) a_{n,0} = O(1)$, then (2.1) holds. This ends the proof of the cases (iv) and (v).

Finally, we prove the case (vi).

Let t_n be a trigonometrical polynomial of Lemma 1 of the present paper. Then for $m \le n$,

$$S_m (t_n; x) = t_m \qquad \text{and} \qquad S_m (f;x) - t_m = S_m (f - t_n; x).$$

Thus

$$T_n (f;x) - \sum_{k=0}^{n} a_{n,k} t_k (x) = \sum_{k=0}^{n} a_{n,k} S_k (f - t_n; x),$$

where

$$S_k (f - t_n; x) = \frac{1}{\pi} \int_0^{2\pi} \{f(x+u) - t_n (x+u)\} \frac{\sin\left(k + \frac{1}{2}\right) u}{2 \sin \frac{u}{2}} du.$$

By general form of Minkowski's inequality we get

$$\left\| T_n (f) - \sum_{k=0}^{n} a_{n,k} t_k \right\|_{L^1} \le \frac{1}{2\pi^2} \int_0^{2\pi} |K_n (u)| \, du \int_0^{2\pi} |f(x+u) - t_n (x+u)| \, dx$$

$$= \frac{1}{2\pi^2} \int_0^{2\pi} |K_n(u)| \int_0^{2\pi} |f(x) - t_n(x)| \, dx = \frac{1}{\pi} \|f - t_n\|_{L^1} \int_0^{2\pi} |K_n(u)| \, du$$

$$= \frac{2}{\pi} \|f - t_n\|_{L^1} \int_0^{\pi} |K_n(u)| \, du = \frac{2}{\pi} \|f - t_n\|_{L^1} \left(\int_0^{\pi/n} |K_n(u)| \, du + \int_{\pi/n}^{\pi} |K_n(u)| \, du \right)$$

$$= \frac{2}{\pi} \|f - t_n\|_{L^1} (I_1 + I_2), \tag{4.3}$$

where

$$K_n(u) = \sum_{k=0}^{n} a_{n,k} \frac{\sin\left(k + \frac{1}{2}\right) u}{2 \sin \frac{u}{2}}.$$

Now, we estimate the quantities I_1 and I_2. By (1.2)

$$I_1 \ll \int_0^{\pi/n} \sum_{k=0}^{n} (k+1) \, a_{n,k} du = O(1). \tag{4.4}$$

If $\left((k+1)^{-\beta} a_{n,k}\right) \in HBVS$, then $\left((k+1)^{-\beta} a_{n,k}\right) \in AMIS$. Hence, for $0 \le l \le m \le n$,

$$K a_{n,m} \ge a_{n,l} \left(\frac{m+1}{l+1}\right)^{\beta} \ge a_{n,l}.$$

Thus $(a_{n,k}) \in AMIS$. Using this and the assumption $(n+1) \, a_{n,n} = O(1)$ we obtain that

$$I_2 \ll a_{nn} \int_{\pi/n}^{\pi} u^{-2} du = O(1). \tag{4.5}$$

Combining (4.3)-(4.5) we have

$$\left\| T_n(f) - \sum_{k=0}^{n} a_{n,k} t_k \right\|_{L^1} \ll \|f - t_n\|_{L^1}. \tag{4.6}$$

Further, by using (4.6) and Lemma 1 for $p = \alpha = 1$, we get

$$\|T_n(f) - f\|_{L^1} \le \left\| T_n(f) - \sum_{k=0}^{n} a_{n,k} t_k \right\|_{L^1} + \left\| \sum_{k=0}^{n} a_{n,k} t_k - f \right\|_{L^1}$$

$$\ll \frac{1}{n} + \left\| \sum_{k=0}^{n} a_{n,k} t_k - f \right\|_{L^1} \le \frac{1}{n} + \sum_{k=0}^{n} a_{n,k} \|t_k - f\|_{L^1} \ll \frac{1}{n} + \sum_{k=0}^{n} (k+1)^{-1} a_{n,k}.$$

By Abel's transformation

$$\|T_n(f) - f\|_{L^1} \ll \frac{1}{n} + \sum_{k=0}^{n-1} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \sum_{i=0}^{k} (i+1)^{\beta - 1}$$

$$+\frac{a_{n,n}}{(n+1)^{\beta}}\sum_{k=0}^{n}(k+1)^{\beta-1} \ll \frac{1}{n}+(n+1)^{\beta}\sum_{k=0}^{n-1}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|+a_{n,n}.$$

Since $\left((k+1)^{-\beta}a_{n,k}\right) \in HBVS$ and $(n+1)\,a_{n,n}=O\,(1)$, then

$$\|T_n\,(f)-f\|_{L^1}=O\left(n^{-1}\right)$$

and (2.1) holds.

This completes the proof of Theorem 5. □

References

[1] P. Chandra, Approximation by Nörlund operators, Mat. Vestnik, 38(1986), 263-269.

[2] P. Chandra, Functions of classes L^p and $Lip\,(\alpha,p)$ and their Riesz means, Riv. Mat. Univ. Parma, (4) 12(1986), 275-282..

[3] P. Chandra, A note on degree of approximation by Nörlund and Riesz operators, Mat. Vestnik, 42(1990), 9-10.

[4] P. Chandra, Trigonometric approximation of functions in L^p-norm, J. Math. Anal. Appl., 275(2002), 13-26.

[5] R. N. Mohapatra and D. C. Russell, Some direct and inverse theorem in approximation of functions, J. Austral. Math. Soc., (Ser. A) 34(1983), 143-154.

[6] M. L. Mittal, B. E. Rhoades, V. N. Mishra and Uaday Singh, Using infinite matrices to approximate functions of class $Lip\,(\alpha,p)$ using trigonometric polynomials, J. Math. Anal. Appl., 326(2007), 667-676.

[7] B. N. Sahney and V. V. Rao, Error bounds in the approximation of functions, Bull. Austral. Math. Soc., 6(1972), 11-18.

[8] E. S. Quade, Trigonometric approximation in the mean, Duke Math. J., 3(1937), 529-542.